Ring structures and mean first passage time in networks

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In this paper we address the problem of the calculation of the mean first passage time on generic graphs. We focus in particular on the mean first passage time on a node s for a random walker starting from a generic, unknown, node x. We introduce an approximate scheme of calculation which maps the original process in a Markov process in the space of the so-called *rings*, described by a transition matrix of size $O(\ln N/\ln \langle k \rangle \times \ln N/\ln \langle k \rangle)$, where N is the size of the graph and $\langle k \rangle$ the average degree in the graph. In this way one has a drastic reduction of degrees of freedom with respect to the size N of the transition matrix of the original process, corresponding to an extremely low computational cost. We first apply the method to the Erdös-Renyi random graphs for which the method allows for almost perfect agreement with numerical simulations. Then we extend the approach to the Barabási-Albert graph, as an example of scale-free graph, for which one obtains excellent results. Finally we test the method with two real-world graphs, Internet and a network of the brain, for which we obtain accurate results.

I. INTRODUCTION

Modern graph theory starts with the study of Erdös-Rényi (ER) random networks in 1959 [1]. In more recent times it has regained a great amount of attention [2] since it has become evident that many different systems can be described as complex scale-free networks, i.e. assemblies of nodes and edges with nontrivial topological properties [3,4].

In this article we focus on the properties of random walks on generic graphs. It is well known that random walk is a fundamental process to explore an environment [5–10], and recently great attention has been devoted to the study of random walk on networks (see, for instance, Refs. [11–20]). In this process a walker, situated on a given node at time t, can be found with probability $1/k$ on any of the k neighbors of that node at time $t+1$.

In particular we are interested in the mean first passage time (MFPT) on a node s for a random walker starting from a generic, unknown, node x. It is important to note here that Noh and Rieger [14] have derived, exploiting the properties of Markov chains, an exact formula for the MFPT T_{sj} of a random walker between two nodes s and j in a generic finite network.

In this paper, however, we do not trivially average T_{sj} over all $j \neq s$, a very costly operation, but we use the concept of *ring* (see also Ref. [21]). In this perspective we study the graph as seen by node s, and partition it in rings according to the topological distance of the different nodes from s (see also Refs. [16,22]). This allows us to map the original Markov problem (of N states) in a Markov chain of drastically reduced dimension $[O(\ln N/\ln \langle k \rangle \times \ln N/\ln \langle k \rangle)]$ and, as a consequence, to calculate MFPT on a generic node s with a reduced computational cost. On the other hand, with the new process, the identity of the single target node s is lost, and all the nodes with the same connectivity (i.e. number of neighbors) are not distinguishable.

Our explicit calculation is almost free of approximations only for Erdös-Rényi random graphs, for which we obtain an excellent agreement between theory and numerical simula-

tions. The more disordered scenario of other complex networks makes the extension of our approach progressively more problematic. Nevertheless we find quite surprisingly that our approach is able to make very good predictions also for other synthetic networks, such as the Barabási-Albert scale-free networks [23], and at least two real-world graphs. In all these cases, the considered networks behave, with respect to the property studied, as if they were random graphs with the same average degree. Finally our approach allows us to show that a random walker recovers rapidly the degree distribution of the network it is exploring.

The paper is organized as follows. In Sec. II the concept of ring is introduced and the Markov process on which the original problem can be mapped is defined. In Sec. III explicit calculations for the case of random graphs are performed. It is shown that the description of a random graph in term of rings is very accurate, and that theoretical predictions for the MFPT are in excellent agreement with results from simulations. Section IV, finally, is devoted to the possible extension of the theory to other networks. Notwithstanding the difficulties that arise in the analytical extension of the theory, it is shown that MFPT of walkers in both artificial and real-life networks can be predicted quite accurately with our theory. It is also shown that a random walk can be used for the reconstruction of the degree distribution of a network.

II. THE NEW PROCESS—RINGS

All the information about a generic graph is contained in the adjacency matrix A whose element $A_{ij}=1$ if nodes i and j are connected, and $A_{ij}=0$ otherwise. We shall consider here only undirected graphs, i.e. $A_{ij}=A_{ji}$, which do not contain any links connecting a node with itself ($A_{ii}=0, \forall i$). The degree of a node, $k(i)$, is given by $k(i)=\sum_j A_{ij}$. Finally, we shall concentrate only on the case of connected graphs, i.e. graphs in which each pair of nodes i, j, with $k(i)$, $k(j) \neq 0$, are connected with at least one single path. From a random walk point of view, the matrix A can be interpreted as the $N \times N$

symmetric transition matrix of the associated Markov process [32].

We are interested in the problem of the average MFPT on a node s of degree $k(s)$ of a random walker that started from a different, unknown, node x. Our idea is mapping the original Markov process A on a much smaller process B that will be asymmetric and will contain self-loops (i.e. $B_{ii} \neq 0$). More precisely we reduce the $N \times N$ matrix to a $O(\ln N/\ln \langle k \rangle \times \ln N/\ln \langle k \rangle)$ matrix.

Given the target node s, we start by subdividing the entire network in subnetworks, or *rings* (see also Refs. [16,21,22]), r_l, with the following property:

$$r_l = \{nodes\ j | d_{sj} = l\}, \tag{1}$$

where $d_{sj} = d_{js}$ is the distance between nodes s and j, i.e. the smallest number of links that a random walker has to pass to get from j to s. These rings will be the states of the new matrix B. Their number, being proportional to the maximum distance between any two nodes in the network, i.e. to the diameter of the network, is $O(\ln N/\ln \langle k \rangle)$ [24,25] (where $\langle k \rangle$ is the average degree of the nodes of the graph).

Other important quantities are the average number $m_{r_l,r_{l+1}} \equiv m_{l,l+1}$ of links that connect all the elements of r_l with all the elements of r_{l+1}, and the average number $m_{l,l}$ of links between nodes belonging to the same r_l. We have trivially $m_{l,l-1} = m_{l-1,l}$ and $m_{l,k} = 0$ if $d_{lk} > 1$.

We now have all the elements to define our new process. We are no longer interested in the exact position of the random walker. The relevant information is now the ring in which the random walker is. The matrix of this process has size $(l_{max} + 1) \times (l_{max} + 1)$, where l_{max} is the diameter of the original graph. The matrix has the following structure [33] (for the case $l_{max} = 6$):

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{10} & b_{11} & b_{12} & 0 & 0 & 0 & 0 \\ 0 & b_{21} & b_{22} & b_{23} & 0 & 0 & 0 \\ 0 & 0 & b_{32} & b_{33} & b_{34} & 0 & 0 \\ 0 & 0 & 0 & b_{43} & b_{44} & b_{45} & 0 \\ 0 & 0 & 0 & 0 & b_{54} & b_{55} & b_{56} \\ 0 & 0 & 0 & 0 & 0 & b_{65} & b_{66} \end{pmatrix}, \tag{2}$$

where $b_{ij} = m_{ij}/(\sum_{k \neq i}^{l_{max}} m_{ik} + 2m_{ii})$ for $i \neq j$, and $b_{ii} = 2m_{ii}/(\sum_{k \neq i}^{l_{max}} m_{ik} + 2m_{ii})$. b_{ij} thus represents the probability of going from ring i to ring j. By definition of rings it is clear that it is not possible to move from a ring to a nonadjacent other ring, while it is obviously possible to move inside a ring, and in this case the number of links must be doubled to take into account that each internal link can be passed in two directions. The elements of the first row of the matrix are set equal to 0 because we are interested in the *first* passage time in the target node s. The probability $P_{ij}^{(t)}$ of going from state i to state j in t steps is given by $(B^t)_{ij}$. If we set $b_{01} = 1$, we would allow the walker to escape from node s, while $b_{00} = 1$ should be used if we were interested in the probability that the walker reached node s before time t. The probability $F_{k(s)}(t)$ that the *first* passage on node s occurs at time t is then

$$F_{k(s)}(t) = \sum_{l=1}^{l_{max}} \frac{n_l}{N-1} (B^t)_{l0}, \tag{3}$$

where n_l is the number of nodes belonging to the ring r_l and each matrix term is weighted with the probability that the random walker started in the ring corresponding to its row, i.e. $n_l/(N-1)$.

The average time MFPT $\tau(k(s))$ can be calculated using Eq. (3) as

$$\tau(k(s)) = \sum_{l=1}^{\infty} l F_{k(s)}(l). \tag{4}$$

III. EXPLICIT CALCULATION FOR RANDOM GRAPHS

A. Static

A random graph is obtained in the following manner: given a finite set of isolated N nodes, all the $N \times (N-1)/2$ pairs of nodes are considered and a link between two nodes is added with probability p. This yields (in the limit $N \to \infty$) to Poisson's distribution for the degree k of a node

$$P(k) = \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}, \tag{5}$$

with $\langle k \rangle = p(N-1)$. It is clear that such a graph does not contain any relevant correlations between nodes and degrees, and this will allow us to obtain exact average relations for the quantities illustrated in the previous section.

The first important quantity to calculate is the average number n_l of elements of r_l. It holds

$$n_{l+1} = \left(N - \sum_{k=0}^{l} n_k\right)[1 - (1-p)^{n_l}], \tag{6}$$

where n_{l+1} is calculated as the expected number of nodes not belonging to any interior ring that are connected with at least a member of r_l. Figure 1 illustrates that Eq. (6) is in excellent agreement with results from simulations. Obviously $n_0 = 1$ and $n_1 = \langle k \rangle$. However, if we know the degree $k(s)$ of node s we can impose $n_1 = k(s)$ and calculate the following $n_{l>1}$ in the usual manner. In fact, this is the way in which we will use Eq. (6).

For the average numbers $m_{l,l+1}$ of links that connect all the elements of r_l with all the elements of r_{l+1}, and $m_{l,l}$ of links between nodes belonging to the same r_l we have

$$m_{l,l+1} = n_l \left(N - \sum_{k=0}^{l} n_k\right) p, \tag{7}$$

$$m_{l,l} = \frac{n_l(n_l - 1)}{2} p,$$

where the fact that a link between two nodes exists with probability p is exploited. As a practical prescription we add that when Eq. (7) yields to nonphysical $m_{l,l} < 0$ one has to redefine $m_{l,l} = 0$. Also these expressions, which are crucial for the construction of the B matrix of Eq. (2), give predic-
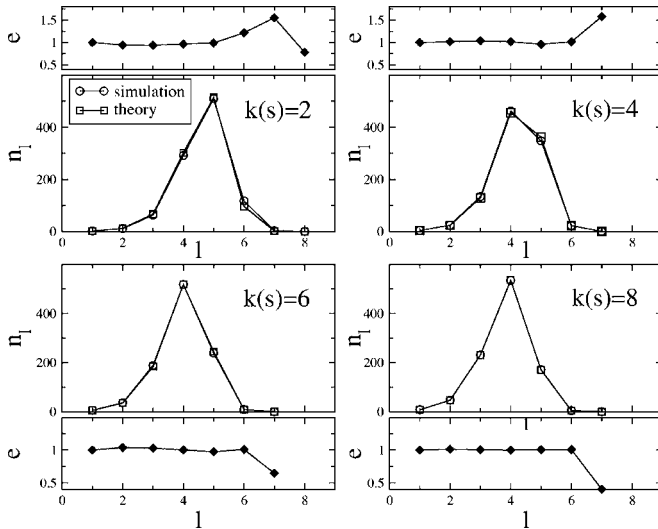
FIG. 1. Nodes n_l per ring: Rings populations n_d are shown in this figure. Comparisons of theoretical previsions from Eq. (6) and data from simulations for different values of $k(s)$ in a single *ER* graph of size $N=10^3$ and $\langle k \rangle=6$ are shown. The fractional error e, defined as the ratio between measured and calculated values, is also plotted for each represented value of $k(s)$. Theoretical average quantities are in excellent agreement with single graph measurements.

tions in excellent agreement with results from simulations (data not shown). As expected we have that, in the limit $p \to 0$, $m_{l,l+1} \to n_{l+1}$, while, when $p \to 1$, $m_{l,l+1} \to n_l \times n_{l+1}$. Note that from Eq. (7) one has that $n_l < 1 \Rightarrow m_{l,l} < 0$, which has clearly no physical meaning.

Before going on it is interesting to make a remark. It is known [22,27] that the nearest neighbors of a generic node have particular properties, e.g., an average degree different from the $\langle k \rangle$ of the graph. Our relations are able to predict this fact. Combining Eqs. (6) and (7) it is easy to see that the average degree of the nodes belonging to r_l *depends* on l, being constant (and larger than $\langle k \rangle$) for low values of l and decreasing rapidly for l large enough. Data from simulation reported in Fig. 2 show that this prediction is correct. This agreement is not surprising since Eqs. (6) and (7) are separately in excellent agreement with simulations, thus being able to predict very accurately the value of the average degree of the nodes belonging to the same ring.

B. Dynamics

So far, we have shown that predictions of the above relations on the static properties of a random graph are correct. We now explore the predictions on the diffusion processes. Figure 3 shows, for different values of the degree $k(s)$ of the target node, the comparison of the predicted values of $F_{k(s)}(t)$, calculated with Eq. (3) and results from simulations. Data from simulations are obtained by selecting randomly, for each of the approximately $N \times N$ runs, one of the nodes of selected degree $k(s)$ present in the network to play the role of s, and one of the remaining $N-1$ nodes to be the starting point of the random walker. The agreement between theory and simulations is very good. The exponential behav-
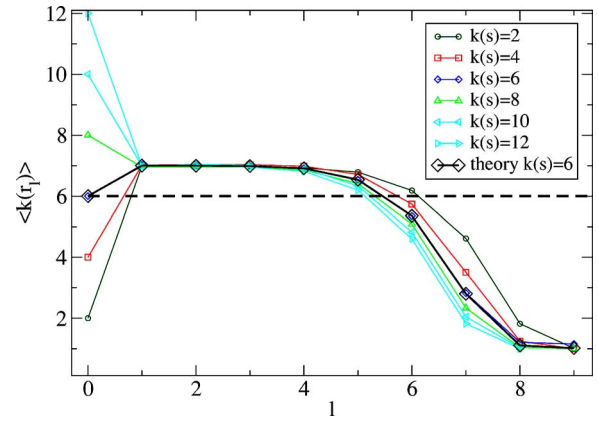

FIG. 2. (Color online) Average ring degree: average degrees of nodes belonging to different rings are shown in function of rings' distances for an *ER* random graph ($N=10^4$ nodes and $\langle k \rangle=6$). A dependence on the ring's distance appears clearly, as predicted by theory. Results for different values of $k(s)$ are shown [predictions shown only for the case $k(s)=6$].

ior, typical of finite ergodic Markov chains, has the form $f(t)=(1/\tau)\exp(-t/\tau)$.

Figure 4 shows that the average MFPT $\tau(k(s))$, calculated using Eq. (4) [with the B matrix built using Eqs. (6) and (7)] are in good agreement (though slightly smaller) with those obtained in simulations. We shall return to the origin of the small disagreement between theory and simulations at the beginning of the next section.

The relation, found both in theory and simulations (see also Ref. [14]), $\tau(k(s))=\tau(1) \times k(s)^{-1}$, can be explained with elementary qualitative probabilistic arguments. In fact, since, as shown in Fig. 2, the average degree of the nodes in r_1 does not depend on $k(s)$ (i.e. on the size n_1 of r_1), also the MFPT on a node of r_1 is independent from n_1. This means that, while the probability of passing from r_1 to s does not depend
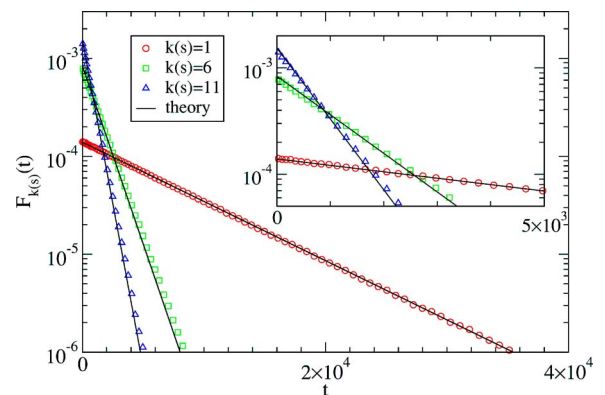

FIG. 3. (Color online) First passage time distributions: FPT probability distributions both measured and calculated for an ER graph with $\langle k \rangle=6$ are presented for different values of $k(s)$. Theoretical predictions are in excellent agreement with data of random walk on a single graph. Theoretical curves are obtained with Eq. (3), and fit the relation $F_{k(s)}(t)=(1/\tau)\exp(-t/\tau)$ with τ obtained with Eq. (4). In the inset the first part of the distribution is show in more detail.
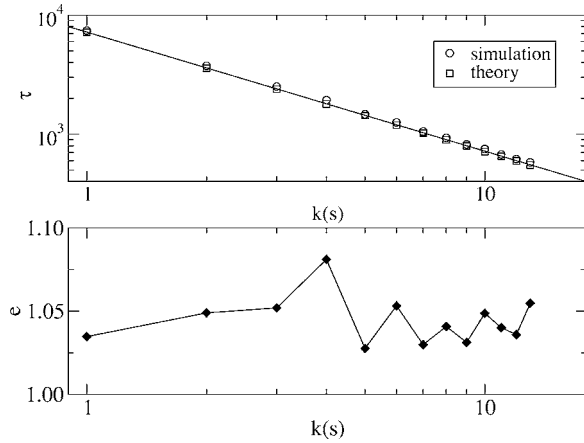
FIG. 4. Mean first passage times: in upper graph, MFPT both measured and calculated [using Eq. (4)] are reported for an ER graph of size $N=10^3$ with $\langle k \rangle = 6$. Error bars on measured values are not visible on the scale of the graph. The line $\tau(k(s)) \simeq \tau(1) \times k(s)^{-1}$ is also plotted. It holds $\tau(1)_{sim} = 7413$ and $\tau(1)_{calc} = 7164$ for values obtained respectively from simulations and from calculation. It can be noted that the order of magnitude of $\tau(1)$ is given by $2M$, where M is the total number of links in the graph; in our case we have $2M \simeq \langle k \rangle \times N = 6000$. In lower graph the fractional error e, defined as the ratio between simulated and calculated MF-PTs, is reported.

on n_1, a larger r_1 is visited more often than a smaller one. Combining these observations, it seems plausible to guess that the MFPT on a target node s with $k(s) > 1$ will be $1/k$ times the MFPT of a node s with $k=1$, and this behavior is indeed observed.

It is worth noting that both the $k(s)^{-1}$ trend and the order of magnitude of $\tau(1)$ can be derived with a simple mean field approach [18,19]. In fact, neglecting all the possible correlations in a graph, the whole random walk process can be approximated with a two-state Markov process where the two states correspond to the walker being at the target node and on any other node. Easy calculations shows that the probability for the walker to arrive at a node s is given by $q(s) = q(k(s)) = k(s)/2M$, where M is the total number of links of the graph. For a fully connected graph this relation gives the exact value of $\tau = \tau(N-1) = N-1$. In a random graph the mean field approach gives better and better results the larger is the mean degree $\langle k \rangle$. For small values of $\langle k \rangle$, only the order of magnitude of $\tau(1)$ is in fact predicted by this approach. The method based on rings, though being less simple, is able to make more accurate predictions for all values of $\langle k \rangle$. Just for comparison we report here data shown in Fig. 4 relative to a network of $N=10^3$ nodes with $\langle k \rangle = 6$: we have $\tau(1)_{sim}/\tau(1)_{calc} = 1.03$ and $\tau(1)_{sim}/2M = 1.24$, where $\tau(1)_{sim}$ and $\tau(1)_{calc}$ are MFPT obtained respectively with simulations and with the ring method.

All the results discussed above allow us to explain the curve presented in Fig. 5, which represents the distribution $P(t)$ of the MFPT in a graph when both s and the starting point of the walker are randomly chosen at each run. $P(t)$ can be calculated here as the convolution of several exponential FPT distributions $F_{k(s)}(t)$ corresponding to the different val-
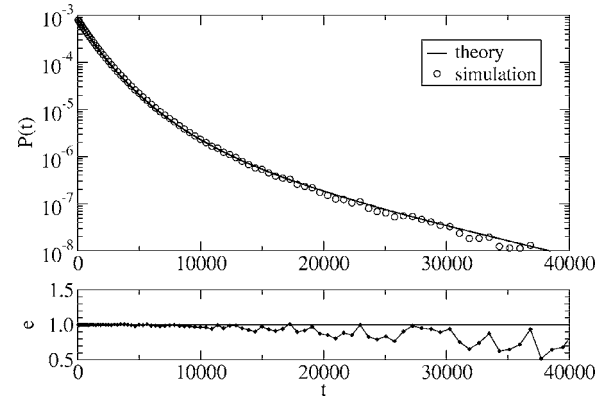

FIG. 5. First passage time distribution—average on all target nodes s: The distribution of the FPT obtained is shown (ER graph: $N=10^3$, $\langle k \rangle = 6$). At each run both the target node s and the starting node of the random walker are randomly chosen. This means that in this distribution the degree of the target nodes is no more fixed. The nonexponential curve is the result of the convolution of several exponential curves obtained for fixed $k(s)$. The theoretical curve, obtained according to Eq. (8), is in excellent agreement with data from simulation performed on a single graph. The fractional error e, defined as the ratio between simulated and calculated data, is also plotted. For large times poor statistics causes bigger fluctuations of e.

ues of $k(s)$, each weighted with the probability of encountering a node of degree $k(s)$ in the graph. More precisely, according to Eq. (5), for each time step t every $F_{k(s)}(t)$ must be weighted with Poisson's weights $c_{k(s)}^{(\langle k \rangle)} = [\langle k \rangle^{k(s)}/k(s)!]e^{-\langle k \rangle}$. We have

$$P(t) = \sum_{j=1}^{\infty} c_j^{(\langle k \rangle)} F_j(t) = \sum_{j=1}^{\infty} c_j^{(\langle k \rangle)} e^{[-t/\tau(j)]} \frac{1}{\tau(j)}. \quad (8)$$

This relation can be written in a more compact way exploiting the fact that $\tau(k(s)) = \tau(1) \times k(s)^{-1}$ [26]. Defined $Z = \langle k \rangle \times \exp[-t/\tau(1)]$, it holds

$$P(t) = \sum_{j=1}^{\infty} Z \frac{Z^{j-1}}{(j-1)!} \frac{e^{-\langle k \rangle}}{\tau(1)} = cZe^Z, \quad (9)$$

where c is the constant $e^{-\langle k \rangle}/\tau(1)$.

IV. EXTENSIONS OF THE THEORY

In the previous sections we have described a method that allows us to calculate the average MFPT on a node s of a walker that started from a generic other node of the graph. We have then obtained exact (average) expressions for the case of random graphs. Unfortunately, the analytical extension of the relations found for this kind of graphs to other graphs (such as, for example, scale-free networks) is difficult. This is due to the fact that Eqs. (6) and (7) exploit the knowledge of the rules according to which a random graph is generated. In other words, the absence of correlations between nodes is the main feature those equations are based on. When correlations are present, the calculation of the number
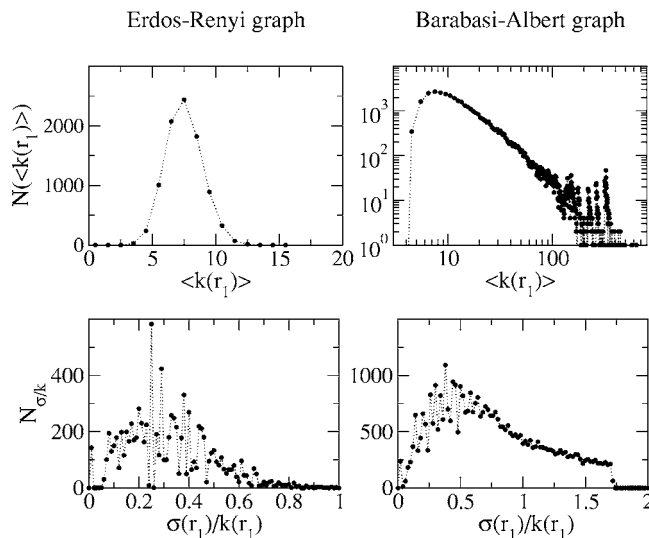
FIG. 6. First ring fluctuations: Fluctuations related to nodes of degree $k=3$ are presented for ER (left) and BA (right) graphs of size $N=1\times10^5$ and $\langle k\rangle=6$. On the two top figures global fluctuations are analyzed. Histograms of the number of nodes vs. the average degree $k(r_1)$ of the nodes of the first ring r_1 for ER graphs (top left) and BA graphs (top right). Logarithmic scale for the ordinate of BA graph must be noted. In the two bottom figures local fluctuations are analyzed. Here the histograms represent the number of rings in function of the ratio $\sigma(r_1)/k(r_1)$, where $\sigma(r_1)$ is the variance of the degree of the nodes belonging to each r_1 for ER graphs (top left) and BA graphs (top right). The higher degree of fluctuations of the BA graphs is evident.

of nodes of the second ring, n_2, is already very difficult (for finite networks) and requires some empirical assumptions [27].

In addition there is a more subtle reason that makes our method difficult to extend. Given the set of all nodes of a graph with a certain degree k, their first rings, although having the same number of nodes, present two kinds of fluctuations. On a global scale, the average degree $k(r_1)$ of the nodes of the first ring does not have a unique value, but in general is distributed according to some probability density. On a local scale, on the other hand, a single ring is not made by identical nodes, and its average degree has a certain variance σ. In Fig. 6 we show global and local fluctuations for both a random graph and a Barabási–Albert (BA) scale-free graph [23]. The preferential attachment rule of the Barabási-Albert network generates a graph with a scale-free form $P(k)\simeq k^{-c}$, with $c=3$, for the degree distribution. As it is evident from Fig. 6, BA graphs have larger fluctuations than random graphs.

With our method of rings, described above, the fluctuations cannot be taken into account. Matrix B (2) is in fact defined under both the assumptions of (i) equivalence of all the nodes with a given k (global homogeneity) and (ii) equivalence of all the nodes inside each ring (local homogeneity). The slight disagreement between our theory and simulation results present in Fig. 4 are thus easily explained in term of local fluctuations of the first ring. In fact, as it is easy to demonstrate using Lagrange multipliers, the assumed
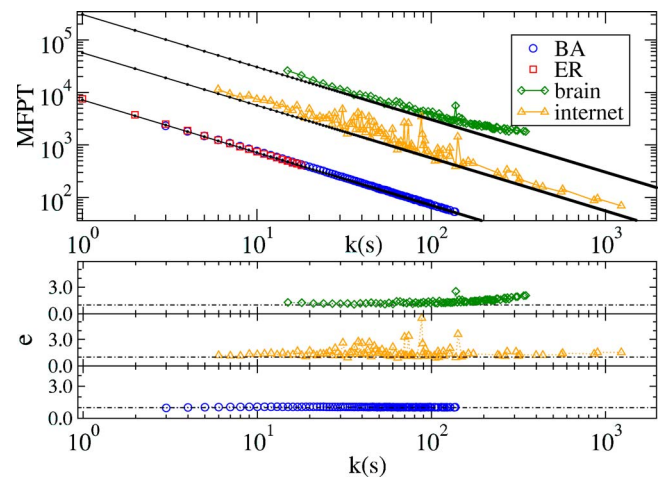

FIG. 7. (Color online) Mean first passage times vs the degree of the target node for different networks. Continuous lines with small filled points (upper graph) are obtained from Eq. (4), i.e. for random ER graphs, while empty symbols of different shapes come from simulations. The agreement between the theory and simulations for both ER graphs (circles) and, less obviously, for the BA graphs, (squares), is evident. Data from real networks are also reported: also in this case the agreement with theoretical predictions is good. Error bars are not visible on the scale of the graph. In the lower graph, fractional errors e, defined as the ratio between measured and calculated values, are reported both for real networks and BA graph. Dashed lines correspond to $e=1$.

local ordered configuration is the most advantageous for a walker that has to reach the node s from the first ring. This is then the reason for which our calculated MFPT are always smaller than those obtained from simulations.

Notwithstanding these difficulties in extending our theory, we found a quite surprisingly result, shown in Fig. 7: given a BA graph with a given average degree $\langle k\rangle$, the average MFPT for a walker starting from a generic node on a node s of degree $k(s)$ is almost equal to the corresponding average MFPT of the same random walk on a random graph with the same average degree. This means that our theory continues to predict very well the MFPT (and hence its exponential distribution). It is remarkable that the theory predicts well also the MFPT on nodes with high degree, which are absent in the corresponding random graph.

The ability of our theory to predict diffusion processes on BA graphs can be due to the modest presence of correlations between its nodes. Many properties of real networks, in fact, are not reproduced by the BA model. One important measure of correlations in a graph is the measure of the average degree of the nearest neighbors (i.e. of the nodes of the first ring) of vertices of degree k, called k_{nn} [28]. While random and BA graphs have a flat k_{nn}, indicating the absence of strong correlations among nodes, many real networks exhibit either an assortative or disassortative behavior. In the first case high-degree vertices tend to be neighbors to high-degree vertices, while in the second case they have a majority of low-degree neighbors. Another important measure of correlation is the clustering coefficient, which is proportional to the probability that two neighbors of a given node are also

neighbor of themselves. Again, BA and random graphs, in which clustering is very poor, do not reproduce the clustering properties of many real networks.

In order to check how far our theory can predict the MFPT on correlated graphs we have performed two sets of experiments on real networks. We have considered in particular a network of Internet at the level of Autonomous Systems [29,30], which exhibits a disassortative mixing feature and a recently proposed scale-free brain functional network [31], which exhibits an assortative mixing feature as well as a strong clustering coefficient.

The results for the MFPT for these two networks, as a function of the degree of the target node, are reported in Fig. 7. Though the agreement between theory and simulation is not perfect, it remains good. In particular we find again the approximate trend $\tau(k(s)) = \tau(1) \times k(s)^{-1}$.

Now we have all the elements to estimate the probability of finding a random walker on a node of a given degree k. On the one hand, in fact, it seems obvious that this probability is related to the fraction $f(k)$ of nodes of that degree in the network, while on the other hand we now know that the MFPT on such a node is proportional, on average, to $1/k$. It is then reasonable to argue that the probability for a random walker being on a node of degree k is proportional to $kf(k)$.

We have tested this hypothesis in an experiment reported in Fig. 8. In the experiment a walker has explored a BA network and an ER random graph with $N = 10^5$ nodes for N time steps. At each time step the degree of the visited node was recorded and the normalized histogram of the fraction of time spent on nodes of any degree is reported in Fig. 8. In the limit of infinite time steps this histogram would indicate exactly the probability of finding the walker of a node of a given k. According to our hypothesis, this histogram should be described by the function $P(k)k/\langle k \rangle$, where $P(k)$ is the degree distribution of the considered network, and the figure shows this is in fact the case, already after a relatively small number of time steps.

Finally, it is worth noting that the previous argument can be reversed. A walker able to record the degree of each node it traverses can be used to determine the degree distribution of the network it travels. In fact, if $t(k)$ is the fraction of time spent on nodes of degree k, it holds that $f(k) \propto t(k)/k$. The average degree $\langle k \rangle$ is then trivially obtained by requiring the normalization of the estimated $P(k)$.

V. CONCLUSIONS

In this paper we addressed the problem of the computation of the mean first passage time on a selected node s of random walkers starting from different nodes on a generic network. We have introduced an approximate method, based on the concept of rings, which maps the original Markov process on another Markov process in a much smaller space. This allows for a drastic reduction of the computational cost. In the case of ER random graphs we have been able to analytically derive all the quantities of interest and we have shown that our method gives predictions, both for static and dynamic properties, in excellent agreement with results found in simulations. Even if this method is promising, ana-
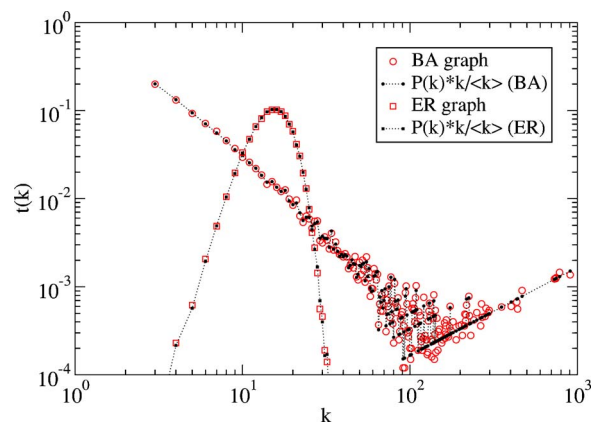

FIG. 8. (Color online) Degree distribution exploration: a random walker explores the degree distribution of two networks, a BA graph of size $N = 10^5$ and $\langle k \rangle = 6$ and an ER random graph of size $N = 10^5$ and $\langle k \rangle = 15$. Simulations in which a walker travels the networks for N time steps have been performed. Empty symbols in figure represent the fraction of time spent on a node of degree k. Filled symbols joined by light lines are obtained with the relation $P(k)k/\langle k \rangle$, where $P(k)$ is the degree distribution of the considered network. Filled points fit well the experimental data, and, as it is obvious, a longer walking process would allow for better agreement. The probability of finding a random walker on a node of a given degree k is related to the degree distribution of the graph via the inverse of $1/k$ MFPT scaling relation (apart from a normalizing factor). For BA graphs it holds $P(k) \propto k^{-3}$. The first part of the curve $P(k)k/\langle k \rangle$ presented in the figure is indeed $c \times k^{-2}$, where c is a constant. The region of higher degrees, on the other hand, grows linearly due to the fact that in a finite-size realization the statistics on high-degree nodes is poor and the degree distribution in this region is flat. We avoided any binning in experimental or theoretical data to make clear that the walker exploration of the network is not perfect.

lytical results are difficult to obtain for nonrandom graphs. However, quite surprisingly, we have found that MFPTs calculated with our theory for ER graphs are in excellent agreement also with simulations of dynamics on BA networks and in good agreement with results obtained with random walkers on two real networks, thus making our method an easy tool to predict MFPT time-related quantities in many cases.

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[32] A random walk can be seen as a Markov process with the identification position-state.

[33] For an easier notation rows (and columns) are labeled with indexes starting from 0, instead of 1.